

Connections between high energy QCD and statistical physics

Stéphane Munier

Centre de physique théorique, École polytechnique, CNRS, 91128 Palaiseau, France

Abstract

It has been proposed that the energy evolution of QCD amplitudes in the high-energy regime falls in the universality class of reaction-diffusion processes. We review the arguments for this correspondence, and we explain how it enables one to compute analytically asymptotic features of QCD amplitudes.

The high-energy regime of QCD has been intensively studied in deep-inelastic e - p scattering at HERA, in heavy-ion collisions at RHIC, and will be probed in proton and heavy-ion scattering at LHC. It is rich of interesting theoretical structures: Links have been found or conjectured with conformal field theory, string theory, and more recently, with a class of models known in statistical physics.

When hadrons scatter at very high energy, the color fields that are generated at the interaction point have a large strength. Perturbative methods qualify as soon as there is a large transverse momentum scale in the event: This property enables one, under certain conditions, to derive QCD evolution equations, in the form of partial differential equations (which can also be stochastic in particular formulations such as the dipole model [1]). On the other hand, strong fields cause the parton densities to saturate, which makes this evolution nonlinear.

Similar looking stochastic nonlinear partial differential equations also appear in problems of apparently different physical origin, such as reaction-diffusion, or population evolution. The goal of this short review is to explain that these similarities are not casual, and that once understood, they can help the derivation of new results for QCD cross sections. We refer the reader to the original papers [2, 3] for the details, and to Ref. [4] for a more extensive review.

In the following, we will consider the scattering of two hadrons, and we will aim at computing their cross section at very high energies. Their relative rapidity is denoted by Y . Since our discussion will rely on resummed perturbative QCD, we think of these hadrons as being small objects, such as color dipoles found for example as fluctuations of highly virtual photons. We will always be discussing a definite region of impact parameter.

1 High energy QCD and reaction-diffusion

Cross sections are measured by counting the number of events that are registered in a detector within a given interval of time. Each single event results from an interaction between the scattering hadrons *realized as definite quantum states*, that is, as particular Fock states. Let us go to the frame in which one hadron is almost at rest, while the other one carries most of the kinetic energy, and thus develops a highly-occupied Fock state. As long as saturation effects are negligible (i.e. far from the unitarity limit in which the hadrons appear black to each other), the probability of interaction is proportional to the number of partons in the fast hadron whose transverse momenta

k match the typical momentum scale of the slow hadron. Let us imagine that k is tunable (it is the case when the slow object is a virtual photon), and that one could actually measure the interaction probability of the slow hadron with a *particular Fock state* of the fast hadron. (In practice, it would require the replication of quantum Fock states, which is impossible). We call $T(k)$ this interaction probability (more precisely, it is the forward elastic scattering amplitude at a fixed impact parameter). T is an unphysical quantity, but it will be important to understand it theoretically. The physical amplitude $A(Y, k)$ is just the average of $T(k)$ over all possible Fock state realizations at the considered rapidity Y : $A = \langle T \rangle$, that is, A is the average over all events that may occur, appropriately weighted by their probabilities at a given rapidity.

Standard quantum field theory calculations, based on the evaluation of Feynman diagrams, would directly lead to the expression of A . However, it turns out that such calculations are extremely hard. Instead, understanding first the main analytical features of the scattering probability T off a single typical Fock state of the fast hadron and then averaging over events is a simpler approach, that has been successful in leading to analytical expressions for the asymptotics of the scattering cross sections.

What is precisely known about T is its evolution with rapidity (or energy), at least in the regime in which $T \ll 1$, that is, away from the unitarity limit. When one increases infinitesimally the rapidity of a hadron that has say n partons in its current Fock state, there is a transition rate to a $n + 1, n + 2, \dots$ -parton Fock state that is computable in perturbative QCD. It may be extracted from the BFKL equation. A direct formulation is the color dipole model [1], in which this transition probability is explicitly computed. When $T \ll 1$, T is a linear function of the number of partons. Roughly, it reads $T(k) \sim \alpha_s^2 n(k)$, where $n(k)$ is the number of gluons that have a transverse momentum of the order of k .

This transition to higher Fock states may be captured by a linear stochastic equation, of the form

$$\partial_{\bar{\alpha}Y} T = \chi(-\partial_{\ln k^2}) T + \alpha_s \sqrt{T} \nu, \quad (1)$$

where $\bar{\alpha} = \alpha_s N_c / \pi$. ν is an appropriate stochastic variable that has zero mean, and variations of order unity when $\bar{\alpha}Y$ is increased by one. $\chi(-\partial_{\ln k^2})$ is the usual BFKL kernel. It describes the branching diffusion of partons, at least in the regime of very high energy in which we are interested in. This means that, when acted on T , it roughly behaves like a diffusion term $\partial_{\ln k^2}^2 T$ supplemented by a growth term T (with appropriate coefficients). The noise term is a consequence of discreteness: It implements the fact that we are considering the evolution of one single Fock state, that contains a definite (discrete) number of partons.

On the other hand, when T becomes of the order of 1, *saturation effects* have to enter in order to tame the growth of the number of partons, for unitarity to be preserved. From the work of Balitsky and Kovchegov (BK), we know that in the mean-field limit in which the noise can be neglected (that is, when $A = T$; This is realized when one of the interacting objects is a very large nucleus), the evolution equation for T reads

$$\partial_{\bar{\alpha}Y} T = \chi(-\partial_{\ln k^2}) T - T^2. \quad (2)$$

Hence we shall propose that the full evolution be described by the following stochastic equation:

$$\partial_{\bar{\alpha}Y} T = \chi(-\partial_{\ln k^2}) T - T^2 + \alpha_s \sqrt{T} \nu. \quad (3)$$

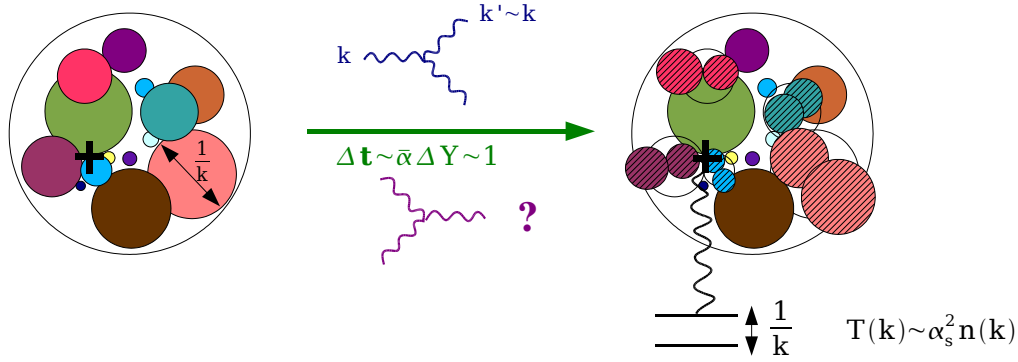


Fig. 1: Sketch of the gluonic content (the gluons are represented by disks) of the hadron in the transverse plane. In a little boost, each gluon can split, or nonlinear effects like recombination can take place. At the time of the interaction, the system is probed by a dipole of size $1/k$, which is sensitive to the number of gluons of similar size at a given impact parameter.

This equation is in the universality class of the stochastic version of the Fisher and Kolmogorov, Petrovsky, Piscounov (F-KPP) equation. (The latter would in fact be obtained by replacing $\chi(-\partial_{\ln k^2})T$ by $\partial_{\ln k^2}T + T$ and \sqrt{T} by $\sqrt{T(1-T)}$; for a review and references, see [5].) A schematic picture of the evolution is presented in Fig. 1.

So far, no one has succeeded in formulating rigorously scattering in QCD in the form of a stochastic equation such as (3), in particular for the way how saturation occurs is not yet fully understood. Is it gluon recombination, as was advocated in the early papers on saturation [6], or some more subtle process? So the best one can do at this stage is to set the noise term in such a way that the evolution of the hadron Fock state by gluon splittings is exactly reproduced away from the unitarity limit, where the nonlinear term may be neglected. A practical implementation of this process would be, for example, Salam's Monte-Carlo code of the dipole model [7] modified by the addition of a suitable saturation condition which makes sure that T (written in coordinate space) keeps always less than 1 and thus that unitarity is preserved. In this procedure, the expression for the noise is unambiguously fixed (it results from the splitting probability of the dipoles), and the BFKL limit is exactly taken into account. Such a procedure was suggested in Ref. [8], but has not been implemented so far for its technical awkwardness. Some other paths were followed: One may alternatively take ν to be a Gaussian white noise¹ [10]. This simple choice enables one to apply the Ito stochastic calculus, and to draw a link with equations established within QCD such as the B-JIMWLK equations (For a review and references, see Ref. [11], and A. Shoshi's talk at this conference [12].) One may also think of the whole process as a reaction-diffusion process, as we will implicitly do in the next section.

All this may look quite arbitrary: We have merely merged two known limits into a single equation, without much further justification. How can one be sure that A obtained from averaging realizations of Eq. (3) looks like the solution of a genuine QCD equation? Although it

¹However, in this case, T should not exactly be the amplitude, but rather a kind of “dual amplitude” – see for example Ref. [9].

might sound strange a priori, the solution to the “arbitrary” equation that we have written down is very likely to contain the exact asymptotics of QCD. This fact is actually related to the universality of the solutions to such evolution equations. The statement is the following: For a large class of processes, i.e. for a number of stochastic functions ν and for a variety of forms of the nonlinearities, the asymptotics of the statistics of T (that is, the physical observables $A \equiv \langle T \rangle$, $\langle T^2 \rangle \dots$) for small α_s and large $\bar{\alpha}Y$ are identical. This is not a theorem, but a conjecture based on a general understanding of how noisy traveling waves propagate. (The propagation mechanism is described in the next section.) The whole point is that the details of the evolution equation do not matter for extracting the asymptotics of the QCD amplitudes.

Let us describe a typical process whose evolution is in the universality class described by the F-KPP equation: reaction-diffusion. This process involves particles on a lattice indexed by some variable x , that evolve by a set of rules of the following form: As time is increased, each particle has a probability either to jump to a nearby site, or to become two particles, or to recombine with another particle on the same lattice site. The balance between creation and recombination of particles determines the equilibrium number of particles on each site N . After a large evolution time, the number of particles on a given site oscillates about N (with an amplitude of the order of the typical statistical fluctuations \sqrt{N}). It is the number of particles per site normalized to N that obeys an equation in the universality class of the F-KPP equation.

At this point, we may establish a simple dictionary between reaction-diffusion and QCD. Time is the evolution variable, so is rapidity: $t \leftrightarrow \bar{\alpha}Y$. The variable in which diffusion takes place is $x \leftrightarrow \ln k^2$. The equilibrium number of particles is $N \leftrightarrow 1/\alpha_s^2$. (In QCD, it is fixed by the unitarity condition² $T \leq 1$.)

2 Statistical methods and application to QCD

In a first step, we ignore the stochastic term, that is, we address the simple BK equation (2), in order to gain intuition on the form of the solutions. A given localized initial condition ($T \sim \alpha_s^2$ in a region of order 1 around some initial scale $\ln k_0^2$: This would be the physical initial condition) will spread and grow under the action of the kernel $\chi(-\partial_{\ln k^2})$, which, as we wrote before, amounts to a branching diffusion. But as soon as T becomes of the order of 1, the nonlinear term enters to compensate the growth, making T saturate. Then further evolution necessarily has the form of two symmetric traveling waves, since the system can only escape to the right and to the left. Let us focus on the rightmoving wave, that travels towards larger values of $\ln k^2$. This wave front is represented schematically in Fig. 2. It turns out that the shape of this wave in its large- $\ln k^2$ tail is exponential, with a slope that is completely fixed by the linear kernel:

$$T \sim e^{-\gamma_0 \ln k^2}, \quad \gamma_0 \text{ being determined by } \frac{\chi(\gamma_0)}{\gamma_0} = \chi'(\gamma_0). \quad (4)$$

²This condition actually holds in coordinate space (when T is a function of transverse sizes). In momentum space, the growth of T with energy is also tamed as soon as the point $T = 1$ has been crossed, although $T(k)$ can take arbitrarily large values. This does not change the conclusions that we shall draw later: The only important feature of the evolution is that T changes behavior in the saturation region. One can see how it goes precisely in QCD e.g. in the numerical simulations presented in Ref. [13].

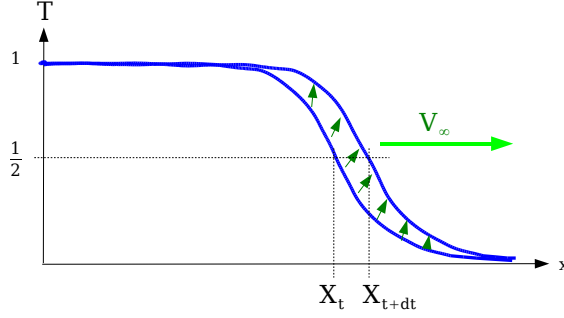


Fig. 2: *Deterministic F-KPP front and its evolution with time.* The arrows show where branching diffusion takes place to drive the motion towards larger values of x .

Since the wave front keeps its shape, it makes sense to characterize its motion by a single velocity V_∞ . The latter is also completely determined by the kernel χ . It reads

$$V_\infty = \frac{\chi(\gamma_0)}{\gamma_0}. \quad (5)$$

In QCD, the position X_t of the wave is called the saturation scale $Q_s(Y)$. It characterizes the momentum below which nonlinear saturation effects (gluon recombination, multiple scattering...) become important. The velocity V_∞ defined above is simply the derivative of $\ln Q_s^2(Y)$ with respect to Y . (Recall that the x -variable is $\ln k^2$.)

Now that we have understood the deterministic limit, we may try to put back the noise. We do not know how to attack the problem directly. Instead, we shall stick to a physical approach, and view the evolution equation (3) as describing a reaction-diffusion process. In this framework, we recall that the origin of the noise was the discreteness of the number of particles on each site. Discreteness means in particular that the number of particles $n(k)$ cannot be a fraction of an integer. Consequently, coming back to the simple-minded relationship³ $T(k) \sim \alpha_s^2 n(k)$, it means that T is either 0 or larger than α_s^2 . Brunet and Derrida [15] proposed to replace the full stochastic equation by a deterministic one that takes into account this basic effect of discreteness, which can easily be done by not allowing any growth when $T < \alpha_s^2$. (It amounts to cutting off the tail of the front; to do this in practice, one may for example replace χ by a modified kernel obtained by subtracting its growth term in the region in which $T < \alpha_s^2$. Note that there is no unique prescription.) The solution to this modified equation is again a traveling wave, that exhibits the same overall exponential decay as given by Eq. (4) (except for an uninteresting additional prefactor). Its velocity now reads⁴

$$V_{BD} = \frac{\chi(\gamma_0)}{\gamma_0} - \frac{\pi^2 \gamma_0 \chi''(\gamma_0)}{2 \ln^2(1/\alpha_s^2)}. \quad (6)$$

³Again, this is not literally true: $T(k)$ is actually continuous, but the tails (below $T = \alpha_s^2$) are decaying exponentially with a characteristic length of one unit in the variable $\ln k^2$. This is steep enough for all our arguments to apply as if $T(k)$ itself were discrete.

⁴This result had already been obtained by Mueller and Shoshi [14]. Actually, the understanding of high-energy scattering as a peculiar reaction-diffusion process emerged from a reinterpretation of their work, in the light of the Brunet-Derrida analytical treatment of traveling waves with a cutoff [15].

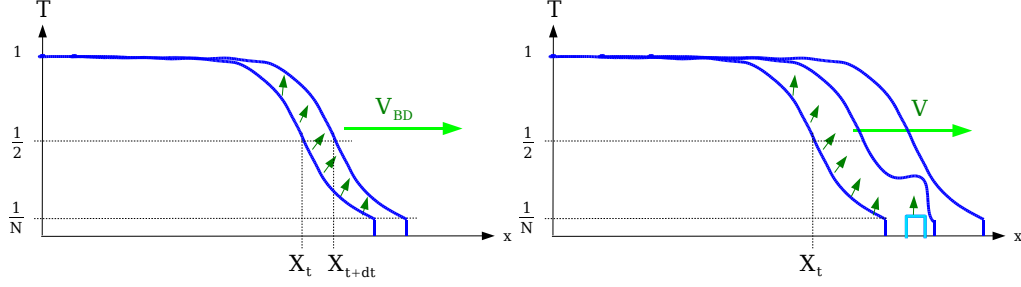


Fig. 3: *Traveling waves solution of the F-KPP equation with a cutoff that simulates discreteness.* In the left sketch, no stochasticity is taken into account while in the right one, particles may randomly be sent ahead of the deterministic front. Their further time/rapidity evolution is also represented.

It is thus less than the velocity of the front in the limit of an infinite number of particles (obtained by letting α_s go to 0), which had to be expected: Indeed, taking into account discreteness amounts to removing some “matter” from the front, which logically slows down its motion.

However, a deterministic solution can only reproduce approximately the realizations of a stochastic evolution. We can incorporate stochasticity back into the picture [16] by noting once again that the noise is only important in the forward tail of the front, where the number of particles is low on the average. From numerical simulations of simple reaction-diffusion models, we observed the following behavior: Most of the time, the motion of the front is almost deterministic, with a velocity given by the solution to the cutoff deterministic equation. From time to time, rarely, a large fluctuation causes a transitory acceleration of the front. This fluctuation consists in one or a few particles being sent far ahead of the deterministic tip of the front, which then evolve into a new front that later gets absorbed by the deterministic front. This behavior is represented in Fig. 3. We conjectured a probability distribution for these fluctuations, as well as the effect that they have on the position of the front after relaxation.

With these elements, we were able to deduce the full statistics of the saturation scale, that is to say not only the mean position (or velocity) of the front,

$$V = \frac{\langle \ln Q_s^2 \rangle}{\bar{\alpha}Y} = \frac{\chi(\gamma_0)}{\gamma_0} - \frac{\pi^2 \gamma_0 \chi''(\gamma_0)}{2 \ln^2(1/\alpha_s^2)} + \pi^2 \gamma_0^2 \chi''(\gamma_0) \frac{3 \ln \ln(1/\alpha_s^2)}{\gamma_0 \ln^3(1/\alpha_s^2)}, \quad (7)$$

but also all its cumulants:

$$\langle \ln^n Q_s^2 \rangle_{\text{cumulant}} = \pi^2 \gamma_0^2 \chi''(\gamma_0) \frac{n! \zeta(n)}{\gamma_0^n} \frac{\bar{\alpha}Y}{\ln^3(1/\alpha_s^2)}, \quad (8)$$

when $n \geq 2$.

Now we recall that the physical amplitude is obtained by averaging T over all possible realizations. Given that the fall off of the large- $\ln k^2$ tail of each single event is exponential, it is not difficult to get the scaling of the scattering amplitude with the help of Eq. (8):

$$A(Y, k) = A \left(\frac{\ln k^2 - \langle \ln Q_s^2(Y) \rangle}{\sqrt{\frac{\bar{\alpha}Y}{\ln^3(1/\alpha_s^2)}}} \right), \quad (9)$$

need to understand more deeply the details of saturation, which probably constrain the form of the noise ν in Eq. (3). Investigations of some possible models have been conducted, sometimes leading to peculiarities in the interpretation, such as negative transition rates [18]. Building a complete picture, valid beyond asymptotics, remains a challenging open question, for which a further breakthrough may be needed.

Finally, our approach to the propagation of noisy traveling waves is not based on a field theory formulation, but is an event-by-event analysis of the shape of realizations, using methods more familiar to statistical physicists than to particle physicists. Being able to recover results such as Eqs. (7), (8), (9) within field theory, starting e.g. from an effective Lagrangian whose building blocks are Reggeon fields, would be a very interesting achievement. Some progress has been made recently, see e.g. Ref. [19].

References

- [1] A. H. Mueller, Nucl. Phys. **B415**, 373 (1994).
- [2] S. Munier and R. Peschanski, Phys. Rev. Lett. **91**, 232001 (2003);
S. Munier and R. Peschanski, Phys. Rev. **D69**, 034008 (2004);
S. Munier and R. Peschanski, Phys. Rev. **D70**, 077503 (2004).
- [3] E. Iancu, A. H. Mueller, and S. Munier, Phys. Lett. **B606**, 342 (2005);
S. Munier, Nucl. Phys. **A755**, 622 (2005).
- [4] S. Munier, Acta Phys. Polon. **B37**, 3451 (2006).
- [5] D. Panja, Physics Reports **393**, 87.
- [6] L. V. Gribov, E. M. Levin, and M. G. Ryskin, Nucl. Phys. **B188**, 555 (1981);
A. H. Mueller and J.-w. Qiu, Nucl. Phys. **B268**, 427 (1986).
- [7] G. P. Salam, Comput. Phys. Commun. **105**, 62 (1997).
- [8] S. Munier, Phys. Rev. **D75**, 034009 (2007).
- [9] C. G. Gardiner, *Handbook of Stochastic Methods*. Springer, 2004.
- [10] E. Iancu and D. N. Triantafyllopoulos, Nucl. Phys. **A756**, 419 (2005).
- [11] H. Weigert, Prog. Part. Nucl. Phys. **55**, 461 (2005).
- [12] A. I. Shoshi (2007). [arXiv:0708.4322 \[hep-ph\]](#).
- [13] R. Enberg, K. J. Golec-Biernat, and S. Munier, Phys. Rev. **D72**, 074021 (2005).
- [14] A. H. Mueller and A. I. Shoshi, Nucl. Phys. **B692**, 175 (2004).
- [15] E. Brunet and B. Derrida, Physical Review E **57**, 2597.
- [16] E. Brunet, B. Derrida, A. H. Mueller, and S. Munier, Phys. Rev. **E73**, 056126 (2006).
- [17] Y. Hatta, E. Iancu, C. Marquet, G. Soyez, and D. N. Triantafyllopoulos, Nucl. Phys. **A773**, 95 (2006);
M. Kozlov, A. I. Shoshi, and B.-W. Xiao, Nucl. Phys. **A792**, 170 (2007);
M. Kozlov, A. Shoshi, and W. Xiang (2007). [arXiv:0707.4142 \[hep-ph\]](#).
- [18] E. Iancu, G. Soyez, and D. N. Triantafyllopoulos, Nucl. Phys. **A768**, 194 (2006);
S. Bondarenko, L. Motyka, A. H. Mueller, A. I. Shoshi, and B. W. Xiao, Eur. Phys. J. **C50**, 593 (2007).
- [19] E. Levin, J. Miller, and A. Prygarin (2007). [arXiv:0706.2944 \[hep-ph\]](#).